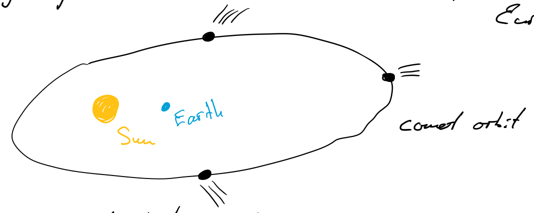
Light momentum

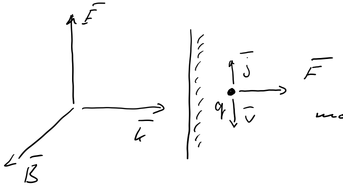
The idea of light momentum was proposed by Kepler (1607) when comet appeared in Europe sky



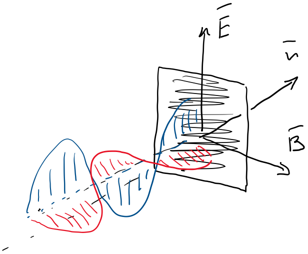
Comet tail points away from the sun regardless the direction in which the comet is travelling.

What is the physics of e/m field carrying momentum?

Recipe: $\underset{\text{force}}{F} \rightarrow \underset{\text{momentum}}{p}$



Due to electric field the charges will oscillate and move along the surface.

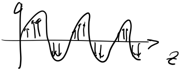


Let's talk about forces that are applied to the charge q:

$$\bar F = q\bar E + \underbrace{\frac{q}{c}[\bar v \bar B]}_{\rightarrow F_L}$$

v - is speed of charge movement

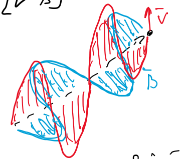
1) $q\bar{E}$ is the only force that makes charge move with $\bar{v}$.

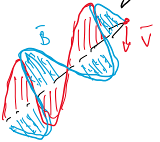


$q\bar{E}$
q
$-q\bar{E}$

$\langle q\bar{E} \rangle = 0$

2) However, exactly the Lorentz force $\frac{q}{c}[\bar{v}\,\bar{B}]$ is the reason of light pressure.





$\bar{E} \parallel \bar{v} \Rightarrow [\bar{v}\,\bar{B}] \Rightarrow \bar{k}$ $\quad [-\bar{v}, -\bar{B}] \Rightarrow \bar{k}$

Let's show it.

Unit vector $\bar{n}$ is the vector of wave propagation direction:

$$\bar{n} = \frac{\bar{k}}{k} \Rightarrow \bar{B} = \frac{1}{c}[\bar{n}\,\bar{E}]$$

$$F_L = q[\bar{v}\,\bar{B}] = \frac{q}{c}\left[v[\bar{n}\,\bar{E}]\right] =$$

$$= q\bar{n}\frac{1}{c}(\bar{v}\,\bar{E})$$

Little mathematical walkaround

$$[\bar{v}[\bar{n}\,\bar{E}]] = \bar{n}(\bar{v}\,\bar{E}) - \bar{E}\cancel{(\bar{v}\cdot\bar{n})}$$

0 since $\bar{v} \perp \bar{n}$

Multiplication of force $q\bar{E}$ to $\bar{v}$ will be equal to power that is given to q by wave.

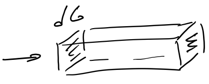
$$P = q\bar{E}\cdot\bar{v} \Rightarrow \boxed{\bar{F}_\sigma = \bar{n}\frac{P}{c}}$$

It is clear now, that the force is directed along the wave propagation.

It is important to notice, that we dropped mathematically rather important parameter – q. That means we can ignore electronic nature of media.

This allows to generalize the theory and connect force directly with light energy density and not specifically with matter.

Let's assume all the energy of incoming light is absorbed on $d\sigma$



Absorbed power:

$$\bar{n}wc = \bar{S} \Rightarrow dP = \bar{S}\,d\bar{\sigma}$$

The corresponding force:

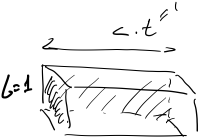
$$d\bar{F} = \frac{\bar{n}}{c}\bar{S}\,d\bar{\sigma} \Rightarrow F = \frac{\bar{n}}{c}\int_\sigma \bar{S}\,d\bar{\sigma}$$

$$\bar{F} = \frac{\bar{n}}{c}(\bar{S}\bar{\sigma}) = w\bar{n}(\bar{n}\bar{\sigma})$$

What does it mean?

The existence of light pressure leads us to believe, that light has momentum.

If we change momentum of a body then the interacting body should carry momentum.



The change in momentum over 1 second is the force $F = \frac{dp}{dt}$

$$\bar{F} = \frac{\bar{S}}{c} = \bar{n} w$$

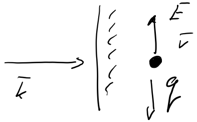
This momentum is distributed in same volume $b \cdot c \cdot t = 1 \cdot c \cdot 1$

Hence volumetric momentum density is

$$\bar{p} = \frac{\bar{S}}{c^2} = \bar{n} \frac{I}{c^2}$$ Maxwell (1873)

Demonstrations

Now let's try to see how Maxwell's equations guide us to determine light momentum.



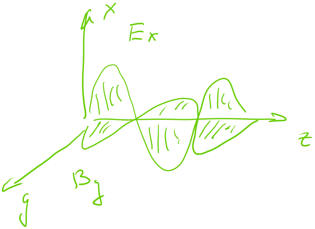
$$F_L = \frac{q}{c}\left[\bar{v}\,\bar{B}\right]$$

$$\begin{cases} \mathrm{rot}\,\bar{E} + \dfrac{\partial \bar{B}}{\partial t} = 0 \\ c^2\,\mathrm{rot}\,\bar{B} - \dfrac{\partial \bar{E}}{\partial t} = \dfrac{j}{\varepsilon_0} \end{cases}$$

Little mathematical walkaround

$$\mathrm{rot}\,\bar{E} = \left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}\right)\bar{e}_x + \left(\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x}\right)\bar{e}_y + \left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right)\bar{e}_z$$

The only non-zero is $\dfrac{\partial E_x}{\partial z}$



$$\mathrm{rot}\,\bar{B} = \left(\frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z}\right)\bar{e}_x + \left(\frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x}\right)\bar{e}_y + \left(\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y}\right)\bar{e}_z$$

The only non-zero is $-\dfrac{\partial B_y}{\partial z}$

Then we can express:

$$\begin{cases} \dfrac{\partial E_x}{\partial z} = -\dfrac{\partial B_y}{\partial t} & \vert \times E_x \\ c^2 \dfrac{\partial B_y}{\partial z} = -\dfrac{\partial E_x}{\partial t} - \dfrac{j}{\varepsilon_0} & \vert \times B_y \end{cases}$$

and sum these equations

$$E_x \frac{\partial E_x}{\partial z} + c^2 B_y \frac{\partial B_y}{\partial z} = -E_x \frac{\partial B_y}{\partial t} - B_y \frac{\partial E_x}{\partial t} - \frac{j B_y}{\varepsilon_0}$$

① However $\bar{j} = nq\bar{v}$

with n - amount of electrons

$\bar{v}$ - speed of electrons

The total Lorentz force on n electrons:

$$\bar{f} = n\bar{F}_L = nq[\bar{v}\,\bar{B}]$$

$$|\bar{f}| = nqvB = jB$$

② $w = \dfrac{\varepsilon_0}{2}(E^2 + c^2 B^2)$ volumetric energy density

Thus we can write:

$$\underbrace{\varepsilon_0 E_x \frac{\partial E_x}{\partial z} + \varepsilon_0 c^2 B_y \frac{\partial B_y}{\partial z}}_{\frac{\partial w}{\partial z}} = \underbrace{-\varepsilon_0 E_x \frac{\partial B_y}{\partial t} - \varepsilon_0 B_y \frac{\partial E_x}{\partial t}}_{-\varepsilon_0 \frac{\partial}{\partial t}(E_x B_y)} - \underbrace{j B_y}_{f}$$

$$\frac{\partial W}{\partial z} = -\varepsilon_0 \frac{\partial}{\partial t}(E_x B_y) - f.$$

If we average over period $T = \frac{2\pi}{\omega}$,

$$\langle E_x B_y \rangle = 0$$

Thus

$$\frac{\partial \tilde{W}}{\partial z} = - \tilde{f}$$

At the same time the pressure of light is:

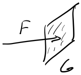
$$P = \int_0^\infty \tilde{f}\, dz$$

Hence

$$P = \int_0^\infty \frac{\partial \tilde{w}}{\partial z}\, dz = \tilde{w} = \frac{I}{c}$$

Since

$$P = \frac{F}{b} \rightarrow \text{pressure}$$



$$F = \frac{dp}{dt} \longrightarrow \text{Newton's second law}$$

$$dp = G\, dV$$

G - volumetric momentum density with $dV = b\,c\,dt$

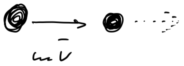
Thus,

$$\tilde{w} = \frac{I}{c} = \frac{G \cdot b\,c\,dt}{b\,dt}$$

$$\Downarrow$$

$$\boxed{G = \frac{I}{c^2}}$$ — Maxwell (1873)

The idea of light momentum brings for the first time the possibility of carpuscular theory of light.



According to Planck:

$W = \hbar\omega$ – energy of photon

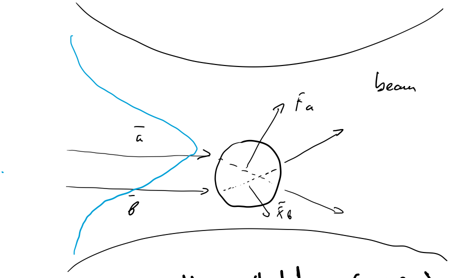
Since $\frac{dW}{dV} = \frac{I}{c} \Rightarrow W = pc$

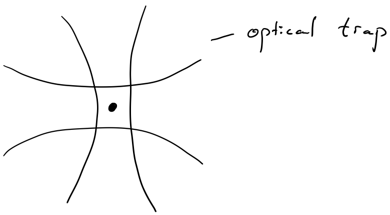
or

$$\boxed{p = \hbar k}$$

Optical trapping

Poynting said, that small value of light pressure allows not to consider it on Earth.



Arthur Ashkin (1986)
Nobel Prize (2018)